ON THE POSSIBILITY OF CONSIDERING THE DEFORMED AND STRESSED STATES OF A MEDIUM AS THE INITIAL STATE

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It is proved in this paper that any state of equilibrium of a medium with non-vanishing stresses and strains may be regarded as the initial state for finite strains and stresses, under a special definition of body forces.

Consider, besides the first initial and the current states of the medium, an intermediate state and regard it as a new second initial state. We assume that in both first and second initial states the medium is in equilibrium but in contrast to the first, in the second initial state there exist strains and stresses. The current state of the medium is also stressed and deformed and corresponds in general to a state of motion. We shall henceforth denote all quantities corresponding to the first initial, second initial and current states of the medium by indices 0, $*$ and $^\vee$, respectively.

Denote by $\varepsilon_{\alpha\beta}{}^\vee$ and $p^{\vee\alpha\beta}$ the components of the tensors of finite strain and stress, measured from the first initial state. The ratios of the components of the stress tensor to the density of the medium $\rho^{-1}p^{\vee\alpha\beta} = \sigma^{\vee\alpha\beta}$ will be called the components of the generalized stress tensor. In nonlinear mechanics it is preferable to consider instead of the stress tensor the generalized stress tensor, since it is known [1] that in the case of elastic deformations the components of the latter possess a potential.

Introduce instead of quantities $\varepsilon_{\alpha\beta}{}^\vee$ and $\sigma^{\vee\alpha\beta}$ new tensorial characteristics of the strains and stresses in the medium in the current state

$\varepsilon_{\alpha\beta}$ and $\sigma^{\alpha\beta}$ which are measured from the second initial state. To this end consider a convected Lagrange coordinate system ξ^1, ξ^2, ξ^3. This system is being deformed together with the medium, and consequently the components of the metric tensor connected with this system vary in time. Denote by t°, t^*, $t^{\vee}$ the instances of time corresponding to the above states of state of the medium and by $\overset{\circ}{g}_{\alpha\beta}$, $\overset{*}{g}_{\alpha\beta}$, $\overset{\vee}{g}_{\alpha\beta}$ the covariant components of the metric tensor in the convected Lagrange coordinate system at these instances. Then the deformation of the medium in the intermediate and current states measured from the first initial state is described by the tensors of finite strain $\overset{*}{\varepsilon}_{\alpha\beta}$ and $\overset{\vee}{\varepsilon}_{\alpha\beta}$ defined by the formulas

$$\overset{*}{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\overset{*}{g}_{\alpha\beta} - \overset{\circ}{g}_{\alpha\beta}), \qquad \overset{\vee}{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\overset{\vee}{g}_{\alpha\beta} - \overset{\circ}{g}_{\alpha\beta}) \tag{1}$$

It is assumed henceforth that the Greek indices run from 1 to 3. The deformation of the medium in the current state measured from the second initial state is characterized by the new tensor of finite strain

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\overset{\vee}{g}_{\alpha\beta} - \overset{*}{g}_{\alpha\beta}) \tag{2}$$

Formulas (1) and (2) imply that quantities $\overset{*}{\varepsilon}_{\alpha\beta}$, $\overset{\vee}{\varepsilon}_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ satisfy the relation

$$\varepsilon_{\alpha\beta} = \overset{\vee}{\varepsilon}_{\alpha\beta} - \overset{*}{\varepsilon}_{\alpha\beta} \tag{3}$$

Let $(1/\rho^*)p^{*\alpha\beta} = \sigma^{*\alpha\beta}$ be the components of the generalized stress tensor in the intermediate state of the medium measured from the first initial state. As the generalized stress tensor in the current state measured from the second initial state and corresponding to the strain tensor $\varepsilon_{\alpha\beta}$ we can consider the tensor

$$\sigma^{\alpha\beta} = \frac{1}{\rho}p^{\alpha\beta} = \frac{1}{\rho}p^{\vee\alpha\beta} - \frac{1}{\rho^*}p^{*\alpha\beta} = \sigma^{\vee\alpha\beta} - \sigma^{*\alpha\beta} \tag{4}$$

Quantities $\varepsilon_{\alpha\beta}$ and $\sigma^{\alpha\beta}$ satisfy the above stated condition, namely in the second initial state they vanish.

Let us now examine the conditions under which the stress tensor $p^{\alpha\beta}$ appears in the equations of motion of the medium in the same way as tensor $p^{\vee\alpha\beta}$. To this end take the equations of motion for the current state of the medium and the equilibrium equations for the intermediate state in the form [1]

$$\rho\sqrt{g^{\vee}}(\mathbf{F} - \mathbf{a}) + \frac{\partial(\sqrt{g^{\vee}}\mathbf{p}^{\beta})}{\partial\xi^{\beta}} = 0, \qquad \rho^*\sqrt{g^*}\mathbf{F}^* + \frac{\partial(\sqrt{g^*}\mathbf{p}^{*\beta})}{\partial\xi^{\beta}} = 0 \tag{5}$$

where $\mathbf{a}$ is the acceleration of a particle, $\mathbf{F}$ the external force per unit mass $\mathbf{p}^{\beta}$ the interior surface force acting on surface with normal $\mathbf{n}_{\beta}$ at instant $t^{\vee}$; $\mathbf{F}^*$ and $\mathbf{p}^{*\beta}$ are the corresponding quantities at instant t^*; $g^{\vee} = |g^{\vee}_{\alpha\beta}|$ and $g^* = |g^*_{\alpha\beta}|$. In formulas (5) and in what follows repeated indices denote summation.

Denote now by $\mathbf{э}_\alpha^*$ and $\mathbf{э}_\alpha^{\vee}$ the vectors of the covariant bases of the Lagrange coordinate system at instances t^* and $t^{\vee}$. The changes of these vectors in passing from one particle of the medium to another are described by the relations [2]

$$\frac{\partial \mathbf{э}_\sigma^*}{\partial \xi^\beta} = \Gamma_{\sigma\ \beta}^{*\gamma} \mathbf{э}_\gamma^*, \qquad \frac{\partial \mathbf{э}_\sigma^{\vee}}{\partial \xi^\beta} = \Gamma^{\vee\ \gamma}_{\ \sigma\beta} \mathbf{э}_\gamma^{*} \tag{6}$$

where $\Gamma^*{}_{\sigma\beta}{}^{\gamma}$ and $\Gamma^{\vee}{}_{\sigma\beta}{}^{\gamma}$ are the Christoffel symbols defined by the formulas

$$\Gamma^{*\ \gamma}_{\ \sigma\beta} = \frac{1}{2} g^{*\gamma\alpha} \left(\frac{\partial g^*_{\alpha\sigma}}{\partial \xi^\beta} + \frac{\partial g^*_{\alpha\beta}}{\partial \xi^\sigma} - \frac{\partial g^*_{\sigma\beta}}{\partial \xi^\alpha} \right), \qquad \Gamma_{\sigma\beta}^{\ \vee\gamma} = \frac{1}{2} g^{\vee\gamma\alpha} \left(\frac{\partial g^{\vee}_{\alpha\sigma}}{\partial \xi^\beta} + \frac{\partial g^{\vee}_{\alpha\beta}}{\partial \xi^\sigma} - \frac{\partial g^{\vee}_{\sigma\beta}}{\partial \xi^\alpha} \right) \tag{7}$$

$g^{*\gamma\alpha}$, $g^{\vee\gamma\alpha}$ being the contravariant components of the metric tensors $g^*_{\alpha\beta}$, $g^{\vee}_{\alpha\beta}$.

Representing the vectors entering equations (5) in the form

$$\mathbf{F} = F^{\vee\alpha} \mathbf{э}^{\vee}_{\ \alpha}, \quad \mathbf{a} = a^{\vee\alpha} \mathbf{э}^{\vee}_{\ \alpha}, \quad \mathbf{p}^\beta = p^{\vee\alpha\beta} \mathbf{э}^{\vee}_{\ \alpha}, \quad \mathbf{F}^* = F^{*\alpha} \mathbf{э}_\alpha^*, \quad \mathbf{p}^* = p^{*\alpha\beta} \mathbf{э}_\alpha^* \tag{8}$$

carrying out the differentiation, and making use of relations (6), we readily observe that each equation (5) is equivalent to three scalar equations

$$\rho \sqrt{g^{\vee}} (\dot{F}^{\vee\alpha} - a^{\vee\alpha}) + \frac{\partial (\sqrt{g^{\vee}} p^{\vee\alpha\beta})}{\partial \xi^\beta} + \sqrt{g^{\vee}}\, \dot{p}^{\vee\sigma\beta} \Gamma^{\vee\alpha}_{\sigma\beta} = 0 \tag{9}$$

$$\rho^* \sqrt{g^*} F^{*\alpha} + \frac{\partial (\sqrt{g^*} p^{*\alpha\beta})}{\partial \xi^\beta} + \sqrt{g^*} p^{*\sigma\beta} \Gamma_{\sigma\ \beta}^{\ *\alpha} = 0 \tag{10}$$

Constructing the difference of equation (9) and (10), and making use of formulas (4) and the continuity equation

$$\rho \sqrt{g^{\vee}} = \rho^* \sqrt{g^*} \tag{11}$$

we arrive at the equations

$$\rho \sqrt{g^{\vee}} (R^\alpha - a^{\vee\alpha}) + \frac{\partial (\sqrt{g^{\vee}} p^{\alpha\beta})}{\partial \xi^\beta} + \sqrt{g^{\vee}} p^{\sigma\beta} \Gamma^{\vee\ \alpha}_{\ \sigma\beta} = 0 \tag{12}$$

where

$$R^\alpha = F^{\vee\alpha} - F^{*\alpha} + \sigma^{*\sigma\beta} (\Gamma^{\vee\ \alpha}_{\ \sigma\beta} - \Gamma^{*\ \alpha}_{\ \sigma\beta}) \tag{13}$$

Equation (12) is analogous to the equations of motion of the medium (9) but it contains the new stress tensor and a new body force.

Thus, if instead of the stress tensor $p^{\vee\alpha\beta}$ we employ the stress tensor $p^{\alpha\beta}$ defined by formulas (4), the equations of motion of the medium remain valid in the same form, provided the body force is defined by (13). Thus, if we use the latter definition the body force vanishes in the second initial state. In general in the current state the body force depends on the internal stresses in the second initial state and on the nature of the subsequent deformation.

Let us express the body force in terms of the characteristics of the displacement of the medium from the second initial state to the current state. To this end let us transform the difference $\Gamma^{\vee}{}^{\alpha}_{\alpha\beta} - \Gamma^{*}{}^{\alpha}_{\sigma\beta}$ in the expression for the force (13). The decompositions of the vectors of one of the bases $\mathfrak{z}_\alpha^*, \mathfrak{z}^\vee{}_\alpha$ in the second basis have the form

$$\mathfrak{z}^\vee{}_\sigma = c^{*}{}^{\omega}_{\sigma}\mathfrak{z}_\omega^*, \qquad \mathfrak{z}_\sigma^* = c^{\vee}{}^{\omega}_{\sigma}\mathfrak{z}^\vee{}_\omega \tag{14}$$

This implies that for the coefficients of the above decomposition we have the relations

$$c^{\vee}{}^{\alpha}_{\omega}c^{*}{}^{\omega}_{\gamma} = \delta^\alpha{}_\gamma, \qquad c^{\vee}{}^{\alpha}_{\omega}c^{*}{}^{\omega}_{\gamma} = \delta^\alpha{}_\gamma \tag{15}$$

The tensors with components $c^{*}{}^{\omega}_{\sigma}$ and $c^{\vee}{}^{\omega}_{\sigma}$ describe an arbitrary displacement of the medium from the second initial state to the current state.

Since transformations (14) are both one-to-one we have

$$|c^{*}{}^{\omega}_{\sigma}| \neq 0, \qquad |c^{\vee}{}^{\omega}_{\sigma}| \neq 0 \tag{16}$$

The quantities $c^{*}{}^{\omega}_{\sigma}$ and $c^{\vee}{}^{\omega}_{\sigma}$ can be expressed in terms of the components of the corresponding displacement vector

$$\mathbf{w} = w^{*\alpha}\mathfrak{z}^*{}_\alpha = w^*{}_\beta\mathfrak{z}^{*\beta} = w^{\vee\sigma}\mathfrak{z}^\vee{}_\alpha = w^\vee{}_\beta\mathfrak{z}^{\vee\beta}$$

where $\mathfrak{z}^{*\beta}$ and $\mathfrak{z}^{\vee\beta}$ are the vectors of the contravariant bases at instances t^* and $t^\vee$ by the formulas [2]

$$c^{*}{}^{\omega}_{\sigma} = \delta^\omega{}_\sigma + \nabla^*{}_\sigma w^{*\omega}, \qquad c^{\vee}{}^{\omega}_{\sigma} = \delta^\omega{}_\sigma - \nabla^\vee{}_\sigma w^{\vee\omega} \tag{17}$$

where

$$\nabla^*{}_\sigma w^{*\omega} = \frac{\partial w^{*\omega}}{\partial \xi^\sigma} + w^{*\lambda}\Gamma^{*}{}^{\omega}_{\lambda\sigma}, \qquad \nabla^\vee{}_\sigma w^{\vee\omega} = \frac{\partial w^{\vee\omega}}{\partial \xi^\sigma} + w^{\vee\lambda}\Gamma^{\vee}{}^{\omega}_{\lambda\sigma} \tag{18}$$

Relations (6) and (14) lead to

$$\frac{\partial c^{*\omega}{}_{\sigma}}{\partial \xi^{\beta}} + c^{*\gamma}{}_{\sigma}\Gamma^{*}{}_{\gamma\beta}^{\omega} = \Gamma^{\vee}{}_{\sigma\beta}^{\gamma}c^{*\omega}{}_{\gamma}, \qquad \frac{\partial c^{\vee\omega}{}_{\sigma}}{\partial \xi^{\beta}} + c^{\vee\gamma}{}_{\sigma}\Gamma^{\vee}{}_{\gamma\beta}^{\omega} = \Gamma^{*}{}_{\sigma\beta}^{\gamma}c^{\vee\omega}{}_{\gamma}$$

By means of the relationships

$$\nabla^{*}{}_{\beta}c^{*\omega}_{\sigma} = \frac{\partial c^{*\omega}{}_{\sigma}}{\partial \xi^{\beta}} + c^{*\gamma}_{\sigma}\Gamma^{*}{}_{\gamma\beta}^{\omega} - c^{*\omega}_{\gamma}\Gamma^{*}{}_{\sigma\beta}^{\gamma}, \qquad \nabla^{\vee}{}_{\beta}c^{\vee\omega}_{\sigma} = \frac{\partial c^{\vee\omega}{}_{\sigma}}{\partial \xi^{\beta}} + c^{\vee\gamma}_{\sigma}\Gamma^{\vee\omega}_{\gamma\beta} - c^{\vee\omega}_{\gamma}\Gamma^{\vee}{}_{\sigma\beta}^{\gamma}$$

and formulas (15) and (17) they can be written in the form

$$\Gamma^{\vee}{}_{\sigma\beta}^{\alpha} - \Gamma^{*}{}_{\sigma\beta}^{\alpha} = c^{\vee\alpha}{}_{\omega}\nabla^{*}{}_{\beta}c^{*\omega}_{\sigma} = (\delta^{\alpha}{}_{\omega} - \nabla^{\vee}{}_{\omega}w^{\vee\alpha})\,\nabla^{*}{}_{\beta}\nabla^{*}{}_{\sigma}w^{*\omega} \tag{19}$$

or

$$\Gamma^{\vee}{}_{\sigma\beta}^{\alpha} - \Gamma^{*}{}_{\sigma\beta}^{\alpha} = -c^{*\alpha}{}_{\omega}\nabla^{\vee}{}_{\beta}c^{\vee\omega}_{\sigma} = (\delta^{\alpha}{}_{\omega} + \nabla^{*}{}_{\omega}w^{*\alpha})\,\nabla^{\vee}{}_{\beta}\nabla^{\vee}{}_{\sigma}w^{\vee\omega} \tag{20}$$

where

$$\nabla^{*}{}_{\beta}\nabla^{*}{}_{\sigma}w^{*\omega} = \frac{\partial^{2}w^{*\omega}}{\partial\xi^{\beta}\partial\xi^{\sigma}} + \frac{\partial w^{*\lambda}}{\partial\xi^{\nu}}(\delta^{\nu}_{\beta}\Gamma^{*}{}_{\lambda\sigma}^{\omega} + \delta^{\nu}_{\sigma}\Gamma^{*}{}_{\lambda\beta}^{\omega} - \delta^{\omega}{}_{\lambda}\Gamma^{*}{}_{\sigma\beta}^{\nu}) + \\ + w^{*\lambda}\left(\frac{\partial\Gamma^{*}{}_{\lambda\sigma}^{\omega}}{\partial\xi^{\beta}} + \Gamma^{*}{}_{\lambda\sigma}^{\tau}\Gamma^{*\omega}_{\tau\beta} - \Gamma^{*\omega}_{\lambda\tau}\Gamma^{*}{}_{\sigma\beta}^{\tau}\right) \tag{21}$$

$$\nabla^{\vee}{}_{\beta}\nabla^{\vee}{}_{\sigma}w^{\vee\omega} = \frac{\partial^{2}w^{\vee\omega}}{\partial\xi^{\beta}\partial\xi^{\sigma}} + \frac{\partial w^{\vee\lambda}}{\partial\xi^{\vee\nu}}(\delta^{\nu}_{\beta}\Gamma^{\nu\omega}_{\lambda\sigma} + \delta^{\nu}_{\sigma}\Gamma^{\vee}{}_{\lambda\beta}^{\omega} - \delta^{\omega}{}_{\lambda}\Gamma^{\vee}{}_{\sigma\beta}^{\nu}) + \\ + w^{\vee\lambda}\left(\frac{\partial\Gamma^{\vee}{}_{\lambda\sigma}^{\omega}}{\partial\xi^{\beta}} + \Gamma^{\vee}{}_{\lambda\sigma}^{\tau}\Gamma^{\vee}{}_{\tau\beta}^{\omega} - \Gamma^{\vee}{}_{\lambda\tau}^{\omega}\Gamma^{\vee}{}_{\sigma\beta}^{\tau}\right) \tag{22}$$

Consequently

$$R^{\alpha} = F^{\vee\alpha} - F^{*\alpha} + \sigma^{*\sigma\beta}\nabla^{*}{}_{\beta}\nabla^{*}{}_{\sigma}w^{*\omega}(\delta^{\alpha}{}_{\omega} - \nabla^{\vee}{}_{\omega}w^{\vee\alpha}) = F^{\vee\alpha} - F^{*\lambda} + \\ + \sigma^{*\sigma\beta}\nabla^{\vee}{}_{\beta}\nabla^{\vee}{}_{\sigma}w^{\vee\omega}(\delta^{\alpha}{}_{\omega} + \nabla^{*}{}_{\omega}w^{*\alpha}) \tag{23}$$

Consider the deformation of the medium from its intermediate state to the current state. This deformation may be described by one of the tensors of finite strain $\varepsilon^{*} = \varepsilon_{\alpha\beta}\partial^{*\alpha}\partial^{*\beta}$ or $\varepsilon^{\vee} = \varepsilon_{\alpha\beta}\partial^{\vee\alpha}\partial^{\vee\beta}$. Tensor ε^{*} is referred to the basis corresponding to the second initial state, while tensor $\varepsilon^{\vee}$ to the basis of the current state.

Assume that the deformation considered is homogeneous. For the condition of homogeneity of the deformation we can take the condition that either tensor ε^{*} or tensor $\varepsilon^{\vee}$ is constant for all particles of the medium. This leads to the relations

$$\nabla^{*}{}_{\gamma}\varepsilon_{\alpha\beta} = \frac{\partial\varepsilon_{\alpha\beta}}{\partial\xi^{\gamma}} - \varepsilon_{\omega\beta}\Gamma^{*}{}_{\alpha\gamma}^{\omega} - \varepsilon_{\alpha\omega}\Gamma^{*}{}_{\beta\gamma}^{\omega} = 0 \tag{24}$$

$$\nabla^{\vee}{}_{\gamma}\varepsilon_{\alpha\beta} = \frac{\partial\varepsilon_{\alpha\beta}}{\partial\xi^{\gamma}} - \varepsilon_{\omega\beta}\Gamma^{\vee}{}_{\alpha\gamma}^{\omega} - \varepsilon_{\alpha\omega}\Gamma^{\vee}{}_{\beta\gamma}^{\omega} = 0 \tag{25}$$

The Lagrange coordinate system at any fixed instance of time can be taken in an arbitrary way. If we assume that it is a rectangular Cartesian system at instant t^*, then conditions (24) take the form

$$\frac{\partial \varepsilon_{\alpha\beta}}{\partial \xi^{\gamma}} = 0 \tag{26}$$

Conditions (25) thereby are not altered. If we now assume that the Lagrange coordinate system is a rectangular Cartesian system at instant $t^{\vee}$, conditions (24) are unchanged whereas conditions (25) take the form (26). Conditions (24) and (25) are not independent: if one is satisfied the other is also satisfied. We now prove this statement.

According to the definition $g^*_{\alpha\beta} = \mathfrak{d}^*_{\alpha} \cdot \mathfrak{d}^*_{\beta}$ and $g^{\vee}_{\alpha\beta} = \mathfrak{d}^{\vee}_{\alpha} \cdot \mathfrak{d}^{\vee}_{\beta}$. Hence, in view of formulas (14) we have

$$g^{\vee}_{\alpha\beta} = c^{*\sigma}_{\alpha} c^{*\tau}_{\beta} g^*_{\sigma\tau}, \qquad g^*_{\alpha\beta} = c^{\vee\sigma}_{\alpha} c^{\vee\tau}_{\beta} g^{\vee}_{\sigma\tau}$$

and therefore for components $\varepsilon_{\alpha\beta}$ we have the expressions

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(c^{*\sigma}_{\alpha} c^{*\tau}_{\beta} - \delta^{\sigma}_{\alpha}\delta^{\tau}_{\beta}) g^*_{\sigma\tau} = \frac{1}{2}(\delta^{\sigma}_{\alpha}\delta^{\tau} \;\; - c^{\vee\sigma}_{\alpha} c^{\vee\tau}_{\beta}) g^{\vee}_{\sigma\beta} \tag{27}$$

In view of formulas (27) and (17) conditions (24) and (25) can be written in the form

$$\nabla^*_{\gamma}\varepsilon_{\alpha\beta} = \frac{1}{2}(c^{*\omega}_{\beta}\nabla^*_{\gamma}\nabla^*_{\alpha}w^*_{\omega} + c^{*\omega}_{\alpha}\nabla^*_{\gamma}\nabla^*_{\beta}w^*_{\omega}) = 0 \tag{28}$$

$$\nabla^{\vee}_{\gamma}\varepsilon_{\alpha\beta} = \frac{1}{2}(c^{\vee\omega}_{\beta}\nabla^{\vee}_{\gamma}\nabla^{\vee}_{\alpha}w^{\vee}_{\omega} + c^{\vee\omega}_{\alpha}\nabla^{\vee}_{\gamma}\nabla^{\vee}_{\beta}w^{\vee}_{\omega}) = 0 \tag{29}$$

Since the motion of the medium takes place in Euclidean space the components $\nabla^*_{\gamma}\nabla^*_{\alpha}w^*_{\omega}$ and $\nabla^{\vee}_{\gamma}\nabla^{\vee}_{\alpha}w^{\vee}_{\omega}$ are symmetric with respect to indices α and γ. Further, it is readily observed that both (28) and (29) contain 18 independent relations. Consequently, relations (28) and (29) can be regarded as two systems, each containing 18 homogeneous algebraic equations for 18 unknown quantities $\nabla^*_{\gamma}\nabla^*_{\alpha}w^*_{\omega}$ in the first system and the same number of unknown quantities $\nabla^{\vee}_{\gamma}\nabla^{\vee}_{\alpha}w^{\vee}_{\omega}$ in the second system. Consider system (28). It is easy to prove that the determinant of the system

$$D = |c^{*\alpha}_{\beta}|^{*} \tag{30}$$

is different from zero in view of (16). Hence, system (28) has only the zero solution

$$\nabla^*_{\alpha}\nabla^*_{\beta}w^*_{\gamma} = 0 \tag{31}$$

It is readily observed that, conversely, relations (31) imply relations (28), i.e. all quantities $\nabla^*_\gamma \varepsilon_{\alpha\beta}$ and all quantities $\nabla^*_\alpha \nabla^*_\beta w^*_\gamma$ vanish simultaneously.

A similar reasoning makes it possible to prove that quantities $\nabla^\vee_\alpha \nabla^\vee_\beta w^\vee_\gamma$ vanish together with $\nabla^\vee_\gamma \varepsilon_{\alpha\beta}$.

Suppose now that conditions (24) are satisfied. It was proved above that they are equivalent to conditions (31). Hence

$$\nabla^*_\alpha \nabla^*_\beta w^{*\omega} = g^{*\omega\gamma} \nabla^*_\alpha \nabla^*_\beta w^*_\gamma = 0 \tag{32}$$

On the basis of the relations

$$c^{*\alpha}_{\ \ \omega} \nabla^\vee_\beta \nabla^\vee_\sigma w^{\vee\omega} = c^{\vee\alpha}_{\ \ \omega} \nabla^*_\beta \nabla^*_\sigma w^{*\omega} \tag{33}$$

following from expressions (19) and (20) and conditions (32) and (16), we infer that

$$\nabla^\vee_\alpha \nabla^\vee_\beta w^{\vee\omega} = 0 \tag{34}$$

and therefore

$$\nabla^\vee_\alpha \nabla^\vee_\beta w^\vee_\gamma = g^\vee_{\gamma\omega} \nabla^\vee_\alpha \nabla^\vee_\beta w^{\vee\omega} = 0 \tag{35}$$

The latter relations are equivalent to conditions (25).

Thus, it has been proved that conditions (25) follow from conditions (24). Similarly we can prove that, conversely, conditions (24) can be derived from (25).

Conditions (24) should be used when the investigation is carried out on the basis of the characteristics referred to the basis of the second initial state, and conditions (25) when the characteristics are referred to the basis of the current state.

When the deformation is homogeneous the components of the body force (23) take, in view of (32) or the equivalent relations (34), the form

$$R^\alpha = F^{\vee\alpha} - F^{*\alpha} \tag{36}$$

i.e. the body force is independent of the initial stresses.

Thus, the dependence of the body force on the internal initial stresses is connected with the nonhomogeneity of the deformation.

Consider now the conditions under which it is legitimate to neglect the dependence of the body force on the internal stresses in the case of a nonhomogeneous deformation and the conditions under which this dependence is relavant.

If the initial stresses are small, in the expression (23) for the body force the terms containing stresses can be neglected. Then formula (23) takes the same form (36) as in the case of a homogeneous deformation.

Let us now assume that the initial stresses are not small and consider the cases when the deformation of the medium from the basis ∂^*_α to the basis $\check{\partial}_\alpha$, is either finite or small.

In the first case, when the deformation is finite the displacements are also finite and it is clear from formulas (23) that the dependence of the body force on the initial stresses is to be taken into account.

Examine now the second case. The deformation is small if the components of the tensor of finite deformation (ε^* or $\varepsilon^\vee$) are small or if the displacements of the particles of the medium and their derivatives with respect to the coordinates are small.

If components $\varepsilon_{\alpha\beta}$ are small, the displacements in general are not small [3], and consequently, in general the conditions of the first case occur.

If now both the displacements and their first derivatives with respect to coordinates are small, then the dependence of the force on the stresses may be neglected. In fact, in view of formulas (21) (or (22)) in this case the quantities $\nabla^*_\beta\nabla^*_\sigma w^{*\omega}$ (or $\nabla^\vee_\beta\nabla^\vee_\sigma w^{\vee\omega}$) are also small. Consequently, in the expression for the force (23) all terms containing stresses are small. Neglecting these terms we can determine the force in accordance with formulas (36). It is readily observed that in this case the deformation of the medium is not much different from the homogeneous deformation.

Thus, the dependence of the body force on the initial stresses has to be taken into account both when the deformation is finite and in the general case of geometrically linear theory. This dependence may be neglected if the initial stressses are small or the deformation of the medium differs little from the homogeneous deformation.

The dynamic equations (12) examined before are valid for continuous media with arbitrary physical properties. We now proceed to prove that for an elastic medium the new tensorial characteristics of the deformations and stresses $\varepsilon_{\alpha\beta}$ and $\sigma^{\alpha\beta}$ measured from the second initial state are connected by the familiar relations of the linear theory of elasticity [1]

$$\sigma^{\alpha\beta} = \frac{\partial U}{\partial \varepsilon_{\alpha\beta}} \tag{37}$$

Here U is the internal energy of unit mass of the medium, corresponding to the new definition of the initial state; it can be represented in the form

$$U = U^{\vee}\,(g^{*}{}_{\sigma\tau} - 2\varepsilon^{*}{}_{\sigma\tau},\ \varepsilon^{*}{}_{\sigma\tau} + \varepsilon_{\sigma\tau},\ S^{\vee}) - \sigma^{*\alpha\beta}\,(\varepsilon^{*}{}_{\alpha\beta} + \varepsilon_{\alpha\beta}) \tag{38}$$

where $U^{\vee}(g^{0}{}_{\sigma\tau},\ \varepsilon^{\vee}{}_{\sigma\tau},\ S^{\vee})$ is the internal energy defined for the original initial state without stresses and deformations. $S^{\vee}$ is the entropy of the medium.

To prove the statement let us fix the initial and intermediate states of the medium and assume that the current state can change. Assume that the state of the medium is determined by the following parameters: the tensor of finite strain with components $\varepsilon^{\vee}{}_{\alpha\beta}$ and the entropy $S^{\vee}$.

Consider the elementary process corresponding to the increments $dS^{\vee}$ and $d\varepsilon^{\vee}{}_{\alpha\beta}$ of the basic parameters. According to the equation of heat flow for a unit mass of the medium we have

$$dQ^{\vee} = dU^{\vee} - \sigma^{\vee\,\alpha\beta}\,d\varepsilon^{\vee}{}_{\alpha\beta} \tag{39}$$

where $dQ^{\vee}$ is the external flow of heat, and the internal energy is a function of the basic parameters $\varepsilon^{\vee}{}_{\sigma\tau}$, $S^{\vee}$.

Since the intermediate state is fixed, the quantities $\varepsilon^{*}{}_{\sigma\tau}$ and $\sigma^{*\sigma\tau}$ can be regarded as constant quantities. Hence, from (3) we have

$$d\varepsilon^{\vee}{}_{\alpha\beta} = d\varepsilon_{\alpha\beta} \tag{40}$$

For reversible processes

$$dQ^{\vee} = T^{\vee}\,dS^{\vee} \tag{41}$$

where $T^{\vee}$ is the absolute temperature of a fixed particle of the medium. Formulas (3), (4) and (39) to (41) lead to the relation

$$dU = \frac{\partial U}{\partial \varepsilon_{\alpha\beta}}\,d\varepsilon_{\alpha\beta} + \frac{\partial U}{\partial S^{\vee}}\,dS^{\vee} = T^{\vee}\,dS^{\vee} + \sigma^{\alpha\beta}\,d\varepsilon_{\alpha\beta} \tag{42}$$

function U being defined by formula (38). The increments $d\varepsilon_{\alpha\beta}$ and $dS^{\vee}$ are arbitrary and independent and, consequently, relations (42) imply besides the definition of temperature, the constitutive equations (37). These results prove the statement.

Thus, the components of the generalized stress tensor $\sigma^{\alpha\beta}$ possess a potential.

Consider now an elastic-plastic medium. Let us establish the conditions under which the inequality

$$(\sigma^{\vee\alpha\beta} - \sigma^{*\alpha\beta})\, d\varepsilon^{p}{}_{\alpha\beta} \geqslant 0 \tag{43}$$

is valid; this inequality is known in the literature as Drucker's postulate [4].

Examine three different states of equilibrium of the medium: the first initial state B°, the second initial state B^* and the current state $B^{\vee}$. State B° is described by zero stresses and strains. States B^* and $B^{\vee}$ have in general internal stresses and strains. We assume that state $B^{\vee}$ is obtained from B^* by an additional deformation described by the tensor of finite strain with components $\varepsilon_{\alpha\beta}$.

In investigating plastic strains the total strain is regarded as the sum of the elastic and plastic strains

$$\varepsilon^*{}_{\alpha\beta} = \varepsilon^{*e}_{\alpha\beta} + \varepsilon^{*p}_{\alpha\beta}, \qquad \varepsilon^{\vee}{}_{\alpha\beta} = \varepsilon^{\vee e}_{\alpha\beta} + \varepsilon^{\vee p}_{\alpha\beta}, \qquad \varepsilon_{\alpha\beta} = \varepsilon^{e}{}_{\alpha\beta} + \varepsilon^{p}{}_{\alpha\beta} \tag{44}$$

the index e corresponding to the elastic and the index p to the plastic strain. The properties of the medium before and after the plastic deformation are in general different. Following [1] we assume that a change of the mechanical and thermal properties of the medium is due to the residual strains and can be described by means of the covariant components of the tensor of residual strain $\varepsilon^{p}{}_{\alpha\beta}$ and parameters $\chi_s (s = 1, \ldots, e)$, connected either with the residual strains or with the loading processes. Quantities χ_s depend on the loading process in general in a functional way. Consequently, we consider the internal energy of the medium as a function of the parameters

$$\begin{aligned} U = U^{\vee}\, [g^{*}{}_{\sigma\tau} - 2(\varepsilon^{*e}_{\sigma\tau} + \varepsilon^{*p}_{\sigma\tau}),\ \varepsilon^{*e}_{\sigma\tau} + \varepsilon^{*p}_{\sigma\tau} + \varepsilon^{e}{}_{\sigma\tau} + \varepsilon^{p}{}_{\sigma\tau},\ S^{\vee},\ \chi^{\vee}{}_{s},\ k_i] - \\ - \sigma^{*\alpha\beta}(\varepsilon^{*e}_{\alpha\beta} + \varepsilon^{*p}_{\alpha\beta} + \varepsilon^{e}{}_{\alpha\beta} + \varepsilon^{p}{}_{\alpha\beta}) \end{aligned} \tag{45}$$

where $k_i (i = 1, \ldots, m)$ are physical constants.

Let us fix the states B° and B^* and consider a process corresponding to small increments of the parameters from state $B^{\vee}$. Making use of the equation of heat flow we obtain

$$dQ^{\vee} = \frac{\partial U}{\partial \varepsilon^{e}{}_{\alpha\beta}}\, d\varepsilon^{e}{}_{\alpha\beta} + \frac{\partial U}{\partial \varepsilon^{p}{}_{\alpha\beta}}\, d\varepsilon^{p}{}_{\alpha\beta} + \frac{\partial U}{\partial S^{\vee}}\, dS^{\vee} + \frac{\partial U}{\partial \chi^{\vee}{}_{s}}\, d\chi^{\vee}{}_{s} - \sigma^{\alpha\beta} d\varepsilon^{e}{}_{\alpha\beta} - \sigma^{\alpha\beta}\, d\varepsilon^{p}{}_{\alpha\beta} \tag{46}$$

Relation (46) is valid for any reversible or irreversible process. Envisage first the reversible process corresponding to unloading from state $B^{\vee}$. We have

$$d\varepsilon^{\vee p}{}_{\alpha\beta}=0, \qquad d\chi^{\vee}{}_{s}=0, \qquad dQ^{\vee}=T^{\vee}dS^{\vee} \tag{47}$$

and (46) and the assumption of independence of increments $d\varepsilon^{e}{}_{\alpha\beta}$ and $dS^{\vee}$ imply equations

$$T^{\vee}=\frac{\partial U}{\partial S^{\vee}}, \qquad \sigma^{\alpha\beta}=\frac{\partial U}{\partial \varepsilon^{e}{}_{\alpha\beta}} \tag{48}$$

Consider now the elementary process corresponding to the plastic loading from state $B^{\vee}$. Here in general

$$d\varepsilon^{\vee p}{}_{\alpha\beta}\neq 0, \qquad d\chi^{\vee}{}_{s}\neq 0, \qquad dQ^{\vee}=T^{\vee}\,dS^{\vee}-dQ^{\vee\prime} \tag{49}$$

where $dQ^{\vee\prime}$ is the non-compensated heat. Relation (46) with conditions (48) and (49) takes the form

$$\sigma^{\alpha\beta}\,d\varepsilon^{p}{}_{\alpha\beta}=dQ^{\vee\prime}+\frac{\partial U}{\partial \varepsilon^{p}{}_{\alpha\beta}}\,d\varepsilon^{p}{}_{\alpha\beta}+\frac{\partial U}{\partial \chi^{\vee}{}_{s}}\,d\chi^{\vee}{}_{s} \tag{50}$$

If function U in state $B^{\vee}$ has a minimum with respect to parameters $\varepsilon^{p}{}_{\alpha\beta}$ and $\chi^{\vee}{}_{s}$ we have

$$\frac{\partial U}{\partial \varepsilon^{p}{}_{\alpha\beta}}\,d\varepsilon^{p}{}_{\alpha\beta}+\frac{\partial U}{\partial \chi^{\vee}{}_{s}}\,d\chi^{\vee}{}_{s}\geqslant 0 \tag{51}$$

Since $\sigma^{\alpha\beta}=\sigma^{\vee\alpha\beta}-\sigma_{*}{}^{\alpha\beta}$ and $dQ^{\vee\prime}\geqslant 0$ it is readily observed that (50) and condition (51) imply inequality (43).

Thus Drucker's postulate follows from the laws of thermodynamics and the assumption that the internal energy of the medium in the considered state has a minimum with respect to the parameters describing the irreversibility of the processes.

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